Indian Statistical Institute
M. Math. II Year
Semestral Examination 2008-2009
Fourier Analysis

Date:08-12-2008 Duration: 3 Hours

Answer all the questions. Maximum mark you can get is 50. Please do check that at least 62 marks are allotted.

Note: Let $\lambda > 0$, p real. Find a relation between $\int dt\ f(\lambda t + p)e^{-ity}$ and $\frac{1}{\lambda}e^{\frac{iyp}{\lambda}} \int dq\ f(q)\ e^{-iq\frac{y}{\lambda}}$ and you can use it freely.

1. Let $\mathcal{H}$ be a Hilbert space with $\mathcal{H}$ = closed linear span $\{x_1, x_2, \ldots\}$. Assume that there are constants $C_1, C_2 > 0$ such that

$$C_1 \sum_k |a_k|^2 \leq \| \sum a_k\ x_k \|^2 \leq C_2 \sum |a_k|^2$$

for all complex sequences $a_1, a_2, \ldots$. Show that there exist $K_1, K_2 > 0$ depending only on C_1, C_2 such that

$$K_1 \|y\|^2 \leq \sum |\langle y, x_n \rangle|^2 \leq K_2 \|y\|^2$$

for all y in $\mathcal{H}$. [3]

2. Let $\varphi \in L^2(R)$ such that $\{\varphi(t-k) : k \in Z\}$ is a Riesz basis for V_0 = closed lin span $\{\varphi(t-k) : k \in Z\}$. For each $\lambda > 0$ put

$$L_\lambda = \text{closed linear span } \{\sqrt{\lambda}\varphi(\lambda t - k) : k \in Z\}.$$

Let $\rho_\lambda : L^2(R) \to L_\lambda$ be the orthogonal projection. If $g \in L^2(R)$ and g has bounded support show that $\rho_\lambda g \to 0$ as $\lambda \to 0$. [4]

3. Let $\psi \in L^2(R^n)$. Define for $a > 0$, $\underset{\sim}{b}$ in R^n

$$\psi_{a,\underset{\sim}{b}}(\underset{\sim}{x}) = \frac{1}{a^{\frac{n}{2}}}\psi\left(\frac{\underset{\sim}{x} - \underset{\sim}{b}}{a}\right).$$

Define $(W_\psi f)(a, \underset{\sim}{b})$ for f in $L^2(R^n)$ by

$$(W_\psi f)(a, \underset{\sim}{b}) = \langle f, \psi_{a,\underset{\sim}{b}} \rangle.$$

If $f \in L^1 \cap L^2$, find an expression for

$$(2\pi)^{-n/2} \int_{R^n} db\, e^{-itb} (W_\psi f)\ (a, b)$$

in terms of $\hat{f}, \hat{\psi}$ and a and prove your claim. [4]

4. (a) If $\psi \in L^1 \cap L^2(R)$ and

$$\int du \frac{|\hat{\psi}(u)|^2}{|u|} < \infty \text{ show that } \int \psi(t)\ dt = 0.$$

[2]

(b) For real a, define $f_a(t) = (1 - at^2)e^{-(\frac{t^2}{2})}$. Show that there is atmost one a such that $\int du \frac{|f_a(u)|^2}{|u|} < \infty$. [1]

(c) Let f_a be as in (b). Show that there exists real a such that

$$\int du\ \frac{|\hat{f}_a(u)|^2}{|u|} < \infty.$$

[3]

5. Let $B_0, B_1, B_2, \ldots$ be the functions given by $B_0 = \chi_{[0,1]}$,

$$B_1 = B_0 * B_0, \quad B_j = B_{j-1} * B_0 \text{ for } j \geq 2.$$

Fix p in $\{0, 1, 2, 3, \ldots\}$ [say $p = 10,000$]. Let V_j = closed lin sp $\{2^{j/2} B_p(2^j t - k) : k \in Z\}$ for j in Z. Show that $\{V_j : j \in Z\}$ is MRA for $L^2(R)$. [6]

6. Let $\psi \in L^2(R)$ be such that

$$\hat{\psi}(y) = \frac{1}{\sqrt{2\pi}} \chi\{\pi \leq |y| \leq 2\pi\} e^{iy/2}.$$

Show that $\{2^{p/2}\psi(2^p t - k) : p, k \in Z\}$ is

(a) orthonormal family in $L^2(R)$. [2]

(b) basis for $L^2(R)$. [3]

7. Let $f : R \to \mathbb{C}$ be any periodic function of period 2π such that $\int_0^{2\pi} |f(x)|^2\, dx < \infty$. Let $\wedge(h) = \int_0^{2\pi} |f(x+h) - f(x)|^2$. If $\wedge(h) \leq C\, h^\alpha$ for some $\alpha > 1$, show that [5]

$$\sum |\hat{f}(k)| < \infty.$$

8. Let $f : R \to \mathbb{C}$ be a bounded L' function with $\int f(x)\, dx = 0$ and there exist constants $k, \epsilon > 0$ such that $|f(x)| \leq \frac{k}{(1+|x|)^{1+\epsilon}}$. Show that there exists constants $\lambda_j \geq 0$, atoms a_j such that

$$f = \sum_1 \lambda_j\, a_j$$

with $\sum \lambda_j$ bounded by a constant depending on k and ϵ. [6]

9. (Prove the sampling theorem) Let $f \in L^2(R)$ such that supp $\hat{f} \subset [-\pi, \pi]$. Then (a) f is a continuous function [1]

(b) $f(t) = \sum_{k \in Z} f(k) \frac{\sin[\pi(t-k)]}{\pi(t-k)}$. [3]

(c) RHS converges uniformly on compact sets to LHS. [2]

10. (a) Fix $t_0 \in R$. Define $f(x) = e^{it_0 x}$. Find the Fourier transform of the tempered distribution f. [1]

(b) Fix $t_1 \in R$. Let $\delta_{t_1} : S(R) \to \mathbb{C}$ be the linear map given by $\delta_{t_1}(g) = g(t_1)$. Find the Fourier transform of the terpered distribution δ_{t_1}. [1]

(c) State Dinis theorem and Jordans theorem for convergence of fourier series of a function f [2]

(d) State Paley-Wiener theorem for $f \in L^2(R)$ and $f = 0$ on $[0, \infty)$ and the converse. [3]

(e) State Paley-Wiener theorem for $f \in L^2(R)$ and $f(x) = 0$ for $|x| \geq A$ for some A and the converse. [3]

(f) State Poisson summation formula for f in $S(R)$. [1]

11. Let $\varphi \in L^\infty(\mathbb{R}^n)$, $K \in L^1(\mathbb{R}^n)$, $\hat{K}(t) \neq 0 \ \forall \ t \in \mathbb{R}^n$, and $\lim_{|x|\to\infty} (K * \phi)(x) = a_0\hat{K}(0)$, then prove that

(a) $\lim_{|x|\to\infty} (f * \phi)(x) = a_0\hat{f}(0) \ \forall \ f \in L^1(\mathbb{R}^n)$. [3]

(b) Further, if φ is a slowly oscillating, then conclude,

$$\lim_{|x|\to\infty} \phi(x) = a_0. \{\text{where } a_0 \text{ as in (a)}\}.$$

[3]